Continued Fractions and Fractional Derivative Viscoelasticity (Feasibility of Theoretical Arguments of Mathematical Analysis on Computer)

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1 Introduction

An elastic body will restore its original shape and size when it is released from a compressed state. This process of relaxation is typically characterized by the decay of its shrinkage. When the body is used as a damper, mild relaxation is a desirable feature in some engineering applications. For such purposes, materials such as silicone gel would be preferable, since they show viscoelasticity for which relaxation proceeds even milder.

It is known that the dynamical behavior of the viscoelastic body is well represented by fractional derivatives; for reviews, see [1, 2, 3]. Denoting the amount of the shrinkage of the body under compression by $x(t)$, the damping force is assumed to be proportional to its fractional derivative $D^{\nu}x$, $\nu \in \mathbb{R}$, $\nu > 0$. Its Fourier transform is proportional to $(i\omega)^{\nu}$, and it has a real part, which would correspond to elasticity, as well as an imaginary part, which would correspond to viscosity. Apart from this phenomenological reasoning, one may wonder why fractional derivatives are related to viscoelasticity on physical grounds.

The most characteristic feature of the solutions $x(t)$ of differential equations involving fractional derivatives is its power-law decay characteristics. In fact, for large t, $x(t)$ behaves like $t^{-\gamma}$, where γ is a positive fractional number. Recall that with the usual viscous damper, the decay is exponential, i.e., $x(t) \sim e^{-\gamma t}$. Under a scale transformation $t \to \lambda t$, $\lambda \in \mathbb{R}$, the exponential decay scales like $e^{-\gamma t} \to e^{-\lambda\gamma t}$, which implies slower or faster decay depending on whether λ is smaller or larger than unity. On the other hand, the power-law decay scales like $t^{-\gamma} \to (\lambda t)^{-\gamma} = \text{const.} \times t^{-\gamma}$, implying the same power-law decay as before. This suggests a fractal structure for the underlying mechanism of viscoelasticity; due to self-similarity of fractals there is no characteristic scale in the viscoelastic materials. In fact, viscoelastic materials are composed of very large molecules, whose complexity would simulate fractal nature under deformation.

Schiessel and Blumen [4, 1] have shown that by forming a nested ladder of the usual spring-dashpot combinations, one can obtain a mechanical model which has fractal like properties. This has been shown in the Laplace transform $X(s)$ of $x(t)$, for some special choice of parameters. Sakakibara [5] has shown that their result can be cast into a closed form for $x(t)$, which shows the power-law decay $x(t) \sim t^{-\gamma}$. It remains to be shown that such a result is a general property of the fractal structure, not necessarily restricted to the special parameter values. We address this question in this paper, and derive the power-law decay under somewhat relaxed conditions on the parameter values.

In the next sections, we begin by giving a review on fractional differential equations, present some examples of mechanical models of viscoelastic damping, and then proceed to our main discussions. We give a detailed review on the theory of continued fractions, for which references seem to be relatively limited. At the end, some examples are presented using the symbolic mathematics software *MATHEMATICA*.

2 Fractional Derivatives

There are several variants of fractional derivatives, and we employ the Riemann-Liouville fractional derivative. The Riemann-Liouville fractional derivative of $x(t)$ is defined by

$$D^{n-\nu}x(t) = D^n D^{-\nu}x(t), \quad n \in \mathbb{N}, \quad 0 < \nu \leq 1,$$

where $D^n x(t) = d^n x(t)/dt^n$ is the usual n-th derivative, and

$$D^{-\nu}x(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} x(\tau) d\tau, \quad 0 < \nu \leq 1.$$

is the Riemann-Liouville fractional integral of $x(t)$; for reviews, see [2, 6]. Note that D^1 is also denoted by D, and $D^{-1}x(t)$ is the usual integral. In contrast to the usual derivatives of integer order, $D^\nu D^\mu \neq D^{\nu+\mu}$ in general.

In order to find solutions to linear differential equations involving fractional derivatives, we need the eigenfunction of D^ν, $\nu > 0$, i.e., the solution of

$$D^\nu x = a x.$$

Recalling that the eigenfunction of D, or D^{-1}, is e^{at}, define

$$E_t(\nu, a) = D^{-\nu} e^{at} = t^\nu e^{at} \gamma^*(\nu, at),$$

where

$$\gamma^*(\nu, t) = \frac{t^{-\nu}}{\Gamma(\nu)} \int_0^\infty \xi^{\nu-1} e^{-\xi} d\xi$$

is the incomplete gamma function [6]. By direct calculations, it can be shown that

$$\begin{aligned} E_t(\nu, a) &= a E_t(\nu+1, a) + \frac{t^\nu}{\Gamma(\nu+1)}, \\ D^\mu E_t(\nu, a) &= E_t(\nu - \mu, a), \quad \mu \in \mathbb{R}. \end{aligned} \tag{1}$$

Note that $E_t(\nu, a)$ may be expressed as $E_t(\nu, a) = t^\nu E_{1,1+\nu}(a t)$ in terms of the Mittag-Leffler function [2]

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha t + \beta)}.$$

Using the properties (1), it is easy to show that

$$e_n(t, a) = \sum_{k-0}^{n-1} a^{n-k-1} E_t\left(-\frac{k}{n}, a^n\right), \quad n \in \mathbb{N},$$

is the eigenfunction of $D^{1/n}$ with eigenvalue a, i.e., it satisfies

$$D^{1/n} e_n(t, a) = a\, e_n(t, a).$$

Note that the eigenfunction $e_n(t, a)$ is singular at the origin.

Let us consider the linear differential equation $P(D, D^\nu)x = 0$, where P is a polynomial of D and D^ν. The key observation to find its solution is that the Laplace transform of $e_n(t, a)$ is given by

$$\int_0^\infty e^{-st} e_n(t, a) dt = \frac{1}{s^{1/n} - a}.$$

Thus, if $\nu = 1/n$, $n \in \mathbb{N}$, then $P(D, D^\nu)$ is a polynomial $P(D^{1/n})$ of $D^{1/n}$ alone, and hence the solution may be expressed as a linear combination of $e_n(t, a)$. If the heighest order of derivatives in $P(D^{1/n})$ is an integer, the solution is shown to be regular even at the origin [6, 3].

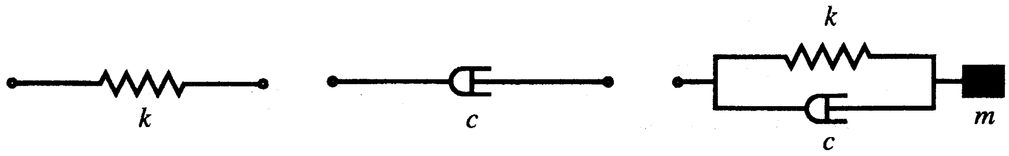


Figure 1: A spring, a dashpot, and the Voigt model

3 Mechanical Models with a Viscoelastic Damper

Let us first consider a usual damped oscillator with a spring and a dashpot (viscous damper) joined in a parallel configuration, called the Voigt model, shown in Figure 1. The equation of motion is given by

$$m D^2 x + c Dx + kx = 0, \tag{2}$$

where m is the mass of the oscillator, c is the viscousity coefficient of the dashpot and k is the spring constant. It is convenient to introduce the natural frequency ω_0 and the damping coefficient ζ by

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{km}}.$$

We also replace $\omega_0 t$ by t, which means that time t is now measured by radians rather than seconds. Then the equation of motion (2) becomes

$$D^2 x + 2\zeta Dx + x = 0. \tag{3}$$

Assuming $0 < \zeta < 1$ for simplicity, the solution corresponding to the initial condition

$$x(0) = 0, \quad Dx(0) = 1 \tag{4}$$

is given by

$$x(t) = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t} \sin\left(\sqrt{1-\zeta^2}\, t\right),$$

which shows the exponential decay, $|x(t)| \sim e^{-\zeta t}$.

We now modify the model by replacing the viscous damper by a viscoelastic damper, with the damping force proportional to $D^\nu x$ in (3). The simplest case is that of $\nu = 1/2$,

$$P(D^{1/2})\, x = D^2 x + \zeta D^{1/2} x + x = 0. \tag{5}$$

The solution corresponding to the initial condition (4) is given by

$$x(t) = \sum_{j=1}^{4} \frac{e_2(t, \alpha_j)}{P'(\alpha_j)},$$

where α_j, $j = 1, 2, 3, 4$, are the roots of the equation $P(z) = 0$, and $P'(z)$ is the derivative of $P(z)$. A detailed study of this solution is given in [3], and it is shown that for large t,

$$x(t) \sim \frac{\zeta}{2\sqrt{\pi}\, t^{3/2}}. \tag{6}$$

Thus, the oscillation damps away following the power-law decay.

In the limit of $m \to 0$ in (3), a model of pure damping, or relaxation is obtained, whose solution is known to show exponential decay. On the other hand, if the term $D^2 x$ is omitted in (5), corresponding to

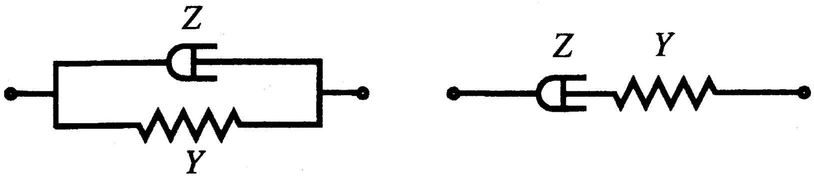


Figure 2: A parallel combination, and a series combination of a spring and a dashpot

the limit of $m \to 0$, the solution becomes singular at the origin, which would not be physically acceptable. In order to overcome this difficulty, an alternative relaxation model

$$Dx + \zeta D^{1/2}x + x = 0$$

has been suggested [7, 8]. The solution with $x(0) = 1$ is shown to obey also the power-law decay (6).

These examples implies that the power-law decay is the characteristic behavior of the fractional derivative viscoelasticity. We now proceed to investigate if power-law decay would emerge from fractal structure.

4 Mechanical Model with Fractal Structure

The force $f(t)$ and the resulting shrinkage $x(t)$ of a spring, shown in Figure 1, is given by

$$f(t) = k\,x(t)$$

where k is the spring constant. In terms of Laplace transform, it reads

$$X(s) = \frac{1}{k}F(s) = Y\,F(s).$$

For a viscous damper, the force $f(t)$ is proportional to the velocity of the shrinkage,

$$f(t) = c\,Dx(t),$$

or

$$X(s) = \frac{1}{cs}F(s) = Z\,F(s).$$

The coefficient of $F(s)$ is the impedance, which we denote by $Z = 1/cs$ and $Y = 1/k$. When the spring and the dashpot are connected in parallel, as in Figure 2, the force F is the sum of the two elements while the shrinkage X is shared in common, and hence the resulting impedance L is given by

$$\frac{1}{L} = \frac{1}{Z} + \frac{1}{Y},$$

When they are connected in series, the shrinkage X is the sum of the two elements while F is shared in common, and hence

$$L = Z + Y.$$

Thus, in the combination shown in Figure 3, the resulting impedance L_n is given by

$$\frac{1}{L_n} = \frac{1}{Z_n} + \frac{1}{Y_{n+1} + L_{n+1}},$$

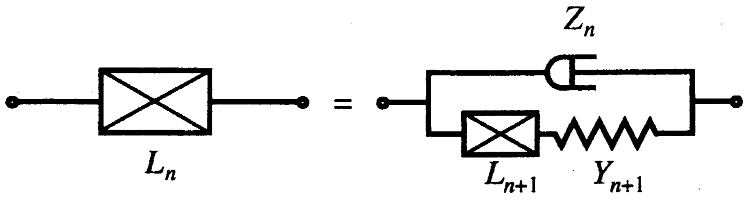


Figure 3: A recursion block of a spring-dashpot loop

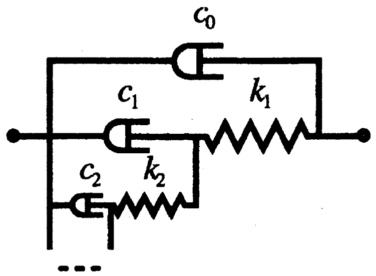


Figure 4: A modified Schiessel-Blumen ladder of spring-dashpot loops

which can be rewritten as

$$L_n = \cfrac{Z_n}{1 + \cfrac{\frac{Z_n}{Y_{n+1}}}{1 + \frac{L_{n+1}}{Y_{n+1}}}}. \tag{7}$$

This is a recursion relation in the form of a continued fraction [9, 10].

We are eventually interested in a modified Schiessel-Blumen ladder in Figure 4, whose impedance will be denoted by L_0. The impulse response is obtained by setting $F(s) = 1$, which corresponds to $f(t) = \delta(t)$. Thus, it is given by the inverse Laplace transform of $L_0(s)$. The impedance $L_0(s)$ is the infinite continued fraction, which is obtained by repeated application of the recursion relation (7), with

$$Z_n = 1/c_n s, \quad Y_n = 1/k_n, \quad n = 1, 2, 3, \ldots. \tag{8}$$

By choosing an appropriate unit of time, we can have $c_0 = 1$ and hence

$$Z_0 = 1/c_0 s = 1/s.$$

We assume that $k_n > 0$ and $c_n > 0$ on physical grounds, but we also discuss the case with $k_n < 0$ and $c_n < 0$ in the last section.

5 Continued Fraction Expansion

A simple example of a continued fraction representation of a fractional number is

$$\frac{233}{177} = 1 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{4 + \cfrac{1}{2}}}} = 1 + \frac{1}{3+}\frac{1}{6+}\frac{1}{4+}\frac{1}{2}.$$

In order to facilitate notational simplicity, the continued fraction is denoted as

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}} = b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\frac{a_3}{b_3 +}\cdots\frac{a_n}{+ b_n} = b_0 + \mathop{\mathrm{K}}_{j=1}^{n}\left(\frac{a_j}{b_j}\right).$$

Denoting

$$\begin{aligned} f_0 &= B_0 = b_0 \\ f_1 &= \frac{A_1}{B_1} = b_0 + \frac{a_1}{b_1} \\ f_2 &= \frac{A_2}{B_2} = b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2} \\ &\vdots \\ f_n &= \frac{A_n}{B_n} = b_0 + \frac{a_1}{b_1 +}\frac{a_2}{b_2 +}\cdots\frac{a_n}{b_n}, \end{aligned} \tag{9}$$

the sequences $\{f_n\}$, $\{A_n\}$ and $\{B_n\}$ are called the n-th approximant, numerator and denominator, respectively.

Continued fractions may be infinite, such as

$$\pi = \frac{4}{1 +}\frac{1^2}{3 +}\frac{2^2}{5 +}\frac{3^2}{7 +}\frac{4^2}{9 +}\cdots, \tag{10a}$$

or, infinitely periodic, such as the Golden ratio,

$$\frac{\sqrt{5}+1}{2} = 1 + \frac{1}{1 +}\frac{1}{1 +}\frac{1}{1 +}\frac{1}{1 +}\frac{1}{1 +}\cdots.$$

Note that the continued fraction representation is not unique. For example, an alternative form of the example of π in (10a) is

$$\pi = \frac{4}{1 +}\frac{1^2/3}{1 +}\frac{2^2/3\cdot 5}{1 +}\frac{3^2/5\cdot 7}{1 +}\frac{4^2/7\cdot 9}{1 +}\cdots. \tag{10b}$$

6 The General Form of Continued Fractions

Consider the map $t_n : \mathbb{C} \to \mathbb{C}$ by

$$t \mapsto t_n(w) = \frac{a_n + c_n w}{b_n + d_n w}, \quad b_n c_n - a_n d_n \neq 0, \quad n = 0, 1, 2, \ldots. \tag{11}$$

These maps are called Möbius transformations. We also define

$$T_0(w) = t_0(w), \quad T_n(w) = T_{n-1}(t_n(w)), \quad n \in \mathbb{N}.$$

We can show that $T_n(w)$ is a Möbius transformation. In fact, we see that

$$T_1(w) = T_0(t_1(w)) = t_0(t_1(w)) = \frac{a_0 + c_0\dfrac{a_1 + c_1 w}{b_1 + d_1 w}}{b_0 + d_0\dfrac{a_1 + c_1 w}{b_1 + d_1 w}} = \frac{(a_0 b_1 + c_0 a_1) + (a_0 d_1 + c_0 c_1)w}{(b_0 b_1 + d_0 a_1) + (b_0 d_1 + d_0 c_1)w}$$

which is a Möbius transformation. Note that the condition

$$(b_0 b_1 + d_0 a_1)(a_0 d_1 + c_0 c_1) - (a_0 b_1 + c_0 a)(b_0 d_1 + d_0 c) = (b_0 c_0 - a_0 d_0)(b_1 c_1 - a_1 d_1) \neq 0$$

is satisfied. Since the successive Möbius transformations is a Möbius transformation, by repeating the process, we see that $T_n(w)$ is a Möbius transformation. Thus, we can write $T_n(w)$ as

$$T_n(w) = \frac{A_n + C_n w}{B_n + D_n w}.$$

Then,

$$T_n(w) = T_{n-1}(t_n(w)) = \frac{(A_{n-1}b_n + C_{n-1}a_n) + (A_{n-1}d_n + C_{n-1}c_n)w}{(B_{n-1}b_n + D_{n-1}a_n) + (B_{n-1}d_n + D_{n-1}c_n)w}$$

from which follows that

$$\begin{aligned}
A_n &= A_{n-1}b_n + C_{n-1}a_n, \\
B_n &= B_{n-1}b_n + D_{n-1}a_n, \\
C_n &= A_{n-1}d_n + C_{n-1}c_n, \\
D_n &= B_{n-1}d_n + D_{n-1}c_n, \\
A_0 &= a_0, \quad B_0 = b_0, \quad C_0 = c_0, \quad D_0 = d_0,
\end{aligned} \tag{12}$$

and

$$B_n C_n - A_n D_n = (b_n c_n - a_n d_n)(B_{n-1}C_{n-1} - A_{n-1}D_{n-1}) = \prod_{k=0}^{n}(b_k c_k - a_k d_k). \tag{13}$$

The inverse relations are obtained by solving (12) as

$$\begin{aligned}
a_n &= \frac{A_{n-1}B_n - A_n B_{n-1}}{A_{n-1}D_{n-1} - B_{n-1}C_{n-1}}, \\
b_n &= \frac{A_{n-1}D_n - B_n C_{n-1}}{A_{n-1}D_{n-1} - B_{n-1}C_{n-1}}.
\end{aligned}$$

The general form of continued fractions is obtained by using the result of the Möbius transformations. To this end, let us define

$$\begin{aligned}
s_0(w) &= b_0 + w, \\
s_n(w) &= \frac{a_n}{b_n + w}, \qquad n = 1, 2, 3, \ldots.
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
S_0(w) &= s_0(w), \\
S_n(w) &= S_{n-1}(s_n(w)), \qquad n = 1, 2, 3, \ldots \\
f_n &= S_n(0), \qquad n = 0, 1, 2, \ldots.
\end{aligned} \tag{15}$$

Note that $s_0(w)$ is a Möbius transformation $t_0(w)$, with $b_0 = 1$, $c_0 = 1$, $d_0 = 0$ and a_0 replaced by b_0 in (11), and that $s_n(w)$ are Möbius transformations $t_n(w)$, with $c_n = 0$, and $d_n = 1$ in (11). Thus, the third and fourth relations in (12) reduce to

$$C_n = A_{n-1}, \quad D_n = B_{n-1}$$

from which the first and second relations in (12) become

$$\begin{aligned} A_n &= A_{n-1}b_n + A_{n-2}a_n, \\ B_n &= B_{n-1}b_n + B_{n-2}a_n. \end{aligned} \tag{16}$$

Thus, we can write

$$S_n(w) = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w},$$

and

$$f_n = S_n(0) = \frac{A_n}{B_n} \tag{17}$$

is the n-th approximant in (9). Note also that (13) reduces to

$$A_n b_{n-1} - B_n A_{n-1} = (-1)^{n-1} \prod_{k=1}^{n} a_k \tag{18}$$

which is called the determinant formula.

With the general form of the continued fractions, we can show the equivalence. Continued fractions $b_0 + \mathrm{K}(a_n/b_n)$ and $\tilde{b}_0 + \mathrm{K}(\tilde{a}_n/\tilde{b}_n)$ are said to be equivalent if their n-th approximants are equal, i.e.,

$$f_n = \tilde{f}_n, \quad n = 0, 1, 2, \ldots. \tag{19}$$

Theorem 1 *Continued fractions are equivalent iff there exists a sequence of non-zero constants* $\{r_n\}$ *with* $r_0 = 1$ *such that*

$$\begin{aligned} \tilde{a}_n &= r_n r_{n-1} a_n, \quad n = 1, 2, 3, \ldots, \\ \tilde{b}_n &= r_n b_n, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{20}$$

Proof Define $\tilde{s}_n(w)$ and $\tilde{S}_n(w)$ as in (14) and (15) with a_n and b_n replaced by $\tilde{a}_n$ and $\tilde{b}_n$, respectively. Then (19) may be rewritten as

$$\tilde{S}_n(0) = S_n(0), \quad n = 0, 1, 2, \ldots. \tag{21}$$

From the second equation in (15), we have

$$s_n(w) = S_{n-1}^{-1}\big(S_n(w)\big), \tag{22}$$

and similarly for $\tilde{s}_n(w)$ and $\tilde{S}_n(w)$.

Now, from (21) with $n = 0$, we have

$$\tilde{b}_0 = \tilde{s}_0(0) = \tilde{S}_0(0) = S_0(0) = s_0(0) = b_0$$

Therefore we can write

$$\tilde{b}_0 = r_0 b_0$$

with $r_0 = 1$. Then, $\tilde{S}_0(r_0 w) = \tilde{s}_0(r_0 w) = \tilde{b}_0 + r_0 w = r_0(b_0 + w) = r_0 s_0(w) = r_0 S_0(w) = S_0(w)$, or

$$\tilde{S}_0^{-1}(w) = r_0 S_0^{-1}(w). \tag{23}$$

Next, from (21) with $n = 1$ and (23),

$$\frac{\tilde{a}_1}{\tilde{b}_1} = \tilde{s}_1(0) = \tilde{S}_0^{-1}\big(\tilde{S}_1(0)\big) = r_0 S_0^{-1}\big(S_1(0)\big) = r_0 s_1(0) = \frac{r_0 a_1}{b_1},$$

which implies that there exists r_1 such that $\tilde{a}_1 = r_1 r_0 a_1$ and $\tilde{b}_1 = r_1 b_1$. It then follows that

$$\tilde{s}_1(r_1 w) = \frac{\tilde{a}_1}{\tilde{b}_1 + r_1 w} = \frac{r_1 r_0 a_1}{r_1 b_1 + r_1 w} = \frac{r_0 a_1}{b_1 + w} = r_0 s_1(w),$$

and

$$\tilde{S}_1(r_1 w) = \tilde{S}_0(\tilde{s}_1(r_1 w)) = \tilde{S}_0(r_0 s_1(w)) = S_0(s_1(w)) = S_1(w),$$

or $\tilde{S}_1^{-1}(w) = r_1 S_1^{-1}(w)$.

Now, assume that there exist r_k such that $\tilde{a}_k = r_k r_{k-1} a_k$, $\tilde{b}_k = r_k b_k$, and

$$\tilde{S}_k^{-1}(w) = r_k S_k^{-1}(w) \tag{24}$$

for $k = 0, 1, 2, \ldots, m$. Then, by (22), (24) and (21),

$$\frac{\tilde{a}_{m+1}}{\tilde{b}_{m+1}} = \tilde{s}_{m+1}(0) = \tilde{S}_m^{-1}\left(\tilde{S}_{m+1}(0)\right) = r_m S_m^{-1}\left(S_{m+1}(0)\right) = r_m s_{m+1}(0) = \frac{r_m a_{m+1}}{b_{m+1}},$$

and hence there exists r_{m+1} such that $\tilde{a}_{m+1} = r_{m+1} r_m a_{m+1}$ and $\tilde{b}_{m+1} = r_{m+1} b_{m+1}$. This proves (20) by induction.

Conversely, assume that (20) hold. Then

$$\tilde{s}_n(r_n w) = \frac{\tilde{a}_n}{\tilde{b}_n + r_n w} = \frac{r_n r_{n-1} a_n}{r_n(b_n + w)} = r_{n-1} s_n(w).$$

Using this result, and noting that $S_n(w) = s_0(s_1(s_2(\cdots s_n(w)\cdots)))$, and similarly for $\tilde{S}_n(w)$, we see that

$$\tilde{S}_n(r_n w) = S_n(w), \quad n = 0, 1, 2, \ldots.$$

This proves (21). □

The alternative representations (10a) and (10b) are related by this equivalence.

7 Continued Fraction Representation of Analytic Functions

The sum

$$L = c_m z^m + c_{m+1} c^{m+1} + c_{m+2} z^{m+2} + \ldots, \quad m \in \mathbb{Z}, \quad c_m \in \mathbb{C}, \quad c_m \neq 0,$$

is called the formal Laurent series (fLs). $L = 0$ is also considered as a fLs. The set $\mathbb{L}$ of all fLs forms a filed. Define $\lambda\colon \mathbb{L} \to \mathbb{R}$ by

$$\lambda(L) = m,$$

and $\lambda(L) = \infty$ for $L = 0$. The following relations follow readily by definition;

$$\begin{aligned} \lambda(L_1 L_1) &= \lambda(L_1) + \lambda(L_2), \\ \lambda(L_1/L_2) &= \lambda(L_1) - \lambda(L_2), \quad L_2 \neq 0 \\ \lambda(L_1 \pm L_2) &= \min\{\lambda(L_1), \lambda(L_2)\}. \end{aligned} \tag{25}$$

If $f(z)$ is a function meromorphic at the origin, then its Laurent expansion will be denoted by $L(f)$, and we denote $\lambda(L(f))$ simply by $\lambda(f)$. A sequence $\{R_n(z)\}$ of functions meromorphic at the origin is said to correspond to a fLs L if

$$\lim_{n\to\infty} \lambda\left(L - L(R_n)\right) = \infty.$$

Every function $f(z)$ meromorphic at the origin has a unique fLs expansion $L(f)$. If $\{R_n(z)\}$ corresponds to an fLs, then the order of correspondence of $R_n(z)$ is defined to be

$$\nu_n = \lambda\big(L - L(R_n)\big).$$

A continued fraction

$$b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \frac{a_3(z)}{b_3(z)} + \cdots$$

is said to correspond to a fLs L if each approximant $f_n(z)$ is a meromorphic function of z at the origin and if $\{f_n(z)\}$ corresponds to L.

Theorem 2 *Let $\{a_n(z)\}$ and $\{b_n(z)\}$ be sequences of functions meromorphic at the origin, with $a_n(z) \not\equiv 0$, $n = 1, 2, 3, \ldots$ and let L_0 be a fLs. Let $\{L_n\}$ be a sequence of fLs defined recursively by*

$$L_{n+1} = \frac{L(a_n)}{L_n - L(b_n)}, \quad n = 0, 1, 2, \ldots, \tag{26}$$

provided

$$L_n \not\equiv L(b_n), \quad n = 0, 1, 2, \ldots. \tag{27}$$

Then the continued fraction

$$b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \frac{a_3(z)}{b_3(z)} + \cdots \tag{28}$$

corresponds to L_0 provided that

$$\begin{aligned} \lambda(b_n) + \lambda(b_{n-1}) &< \lambda(a_n), \quad n = 1, 2, 3, \ldots, \\ \lambda(L_n) + \lambda(b_{n-1}) &< \lambda(a_n), \quad n = 1, 2, 3, \ldots. \end{aligned} \tag{29}$$

Proof Suppose that (27) holds, and let $A_n(z)$, $B_n(z)$ and $f_n(z) = A_n(z)/B_n(z)$ denote the nth numerator, denominator and approximant, respectively, of (28). From (26) we have

$$L_n = L(b_n) + \frac{L(a_{n+1})}{L_{n+1}}, \quad n = 0, 1, 2, \ldots,$$

and hence

$$L_0 = L(b_0) + \frac{L(a_1)}{L(b_1)} + \frac{L(a_2)}{L(b_2)} + \cdots + \frac{L(a_n)}{L_n}, \quad n = 1, 2, 3, \ldots.$$

Since it is equal to $S_n(0)$ in which $L(b_n)$ is replaced by L_n in (17), it follows from (16) that

$$L_0 = \frac{L(a_n)L(A_{n-2}) + L_n L(A_{n-1})}{L(a_n)L(B_{n-2}) + L_n L(B_{n-1})}, \quad n = 2, 3, 4, \ldots.$$

Using (18), we have

$$L_0 - L(f_{n-1}) = L_0 - \frac{L(A_{n-1})}{L(B_{n-1})} = \frac{(-1)^{n-1}\prod_{k=1}^{n} L(a_k)}{L(B_{n-1})\big[L(a_n)L(B_{n-2}) + L_n L(B_{n-1})\big]}, \quad n = 2, 3, 4, \ldots.$$

Under the assumption (29), from the recursion relation (16) we have

$$\lambda(B_0) = 0, \quad \lambda(B_n) = \sum_{k=1}^{n} \lambda(b_k), \quad n = 1, 2, 3, \ldots,$$

and hence

$$\lambda\big(L(a_n)L(B_{n-2}) + L_nL(B_{n-1})\big) = \lambda(L_n) + \sum_{k=1}^{n-1}\lambda(b_k), \quad n = 2, 3, 4, \ldots.$$

Thus, we finally have

$$\begin{aligned}\lambda\big(L_0 - L(f_{n-1})\big) &= \sum_{k=1}^{n}\lambda(a_k) - 2\sum_{k=1}^{n-1}\lambda(b_k) - \lambda(L_n)\\ &= \lambda(a_1) - \lambda(b_1) + \sum_{k=2}^{n-1}\big[\lambda(a_k) - \lambda(b_k) - \lambda(b_{k-1})\big] + \lambda(a_n) - \lambda(b_{n-1}) - \lambda(L_n), \quad n = 2, 3, 4, \ldots.\end{aligned}$$

Due to (29), each term in the summation and the term following it is a positive interger. Hence

$$\lim_{n\to\infty}\lambda\big(L_0 - L(f_{n-1})\big) = \infty$$

proving that (26) corresponds to L_0. □

Sequences that satisfy a system of three-term recurrence relations may be related to continued fractions. In fact, we have the following theorems.

Theorem 3 *Let $\{a_n(z)\}$ and $\{b_n(z)\}$ be sequences of functions meromorphic at the origin, with $a_n(z) \not\equiv 0$, $n = 1, 2, 3, \ldots$, and let $\{P_n\}$ be a sequence of non-zero fLs satisfying the three-term recurrence relations*

$$P_n = L(b_n)P_{n+1} + L(a_{n+1})P_{n+2}, \quad n = 0, 1, 2, \ldots. \tag{30}$$

Then

$$b_0(z) + \frac{a_1(z)}{b_1(z)+}\frac{a_2(z)}{b_2(z)+}\frac{a_3(z)}{b_3(z)+}\cdots$$

corresponds to

$$L = \frac{P_0}{P_1}$$

provided

$$\begin{aligned}\lambda(b_n) + \lambda(b_{n-1}) &< \lambda(a_n),\\ \lambda(P_n/P_{n+1}) + \lambda(b_n) &< \lambda(a_n), \quad n = 1, 2, 3, \ldots.\end{aligned}$$

Proof Let $L_n = P_n/P_{n+1}$. Then (30) reduces to

$$L_n - L(b_n) = \frac{L(a_{n+1})}{L_{n+1}}, \quad n = 0, 1, 2, \ldots.$$

Due to the condition $a_n(z) \not\equiv 0$ Theorem 2 applies. This completes the proof. □

The continued fraction of the form $1 + \mathrm{K}(a_nz/1)$ is called the regular C-fraction. The following theorems are relevant for our purposes.

Theorem 4 *Let $1 + K(a_nz/1)$ be a regular C-fraction such that*

$$\lim_{n\to\infty} a_n = 0 \quad (a_n \neq 0). \tag{31}$$

Then,

(A) The C-fraction converges to a meromorphic function $f(z)$.
(B) The convergence is uniform on every compact subset K of $\mathbb{C}$ which contains no poles of $f(z)$.
(C) $f(z)$ is holomorphic at $z = 0$, and $f(0) = 1$.

Theorem 5 *Let* $1 + K(a_n z/1)$ *be a regular C-fraction such that*

$$\lim_{n\to\infty} a_n = a \neq 0 \tag{32}$$

where a is a complex constant, and let

$$R_a = \left\{z\colon \left|\arg\left(az + \tfrac{1}{4}\right)\right| < \pi\right\}.$$

Then,

(A) The C-fraction converges to a function $f(z)$ *meromorphic in* R_a.
(B) The convergence is uniform on every compact subset K of R_a *which contains no poles of* $f(z)$.
(C) $f(z)$ *is holomorphic at* $z = 0$.

The proofs are length and we refer the reader to the literature [9].

8 Hypergoemetric Functions

As direct consequences of Theorems 4 and 5, we have the continued fraction representation of the hypergeometric function, and the confluent hypergeometric function.

The hypergeometric function is defined by

$$F(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \ldots, \tag{33}$$

where a, b, c are complex constants and $n \notin [0, -1, -2, \ldots]$. It is also denoted as ${}_2F_1(a, b; c; z)$.

Theorem 6 *Let* $\{a_n\}$ *be a sequence of complex numbers defined by*

$$\begin{aligned} a_{2n+1} &= -\frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)}, \quad n = 0, 1, 2, \ldots, \\ a_{2n} &= -\frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)}, \quad n = 1, 2, 3, \ldots, \end{aligned} \tag{34}$$

where a, b, c are constants such that

$$a_n \neq 0, \quad n = 1, 2, 3, \ldots. \tag{35}$$

Then:

(A) The regular C-fraction $1 + \mathrm{K}(a_n z/1)$ *converges to a function* $f(z)$ *meromorphic in the domain*

$$D = \{z\colon 0 < \arg(z-1) < 2\pi\}. \tag{36}$$

(B) The convergence is uniform on every compact subset of D which contains no poles of $f(z)$.
(C) $f(z)$ *is holomorphic at* $z = 0$, *and* $f(0) = 1$.
(D) For all z such that $|z| < 1$,

$$f(z) = \frac{F(a, b; c; z)}{F(a, b+1; c+1; z)}, \tag{37}$$

and hence $f(z)$ *provides the analytic continuation of the function on the right-hand side of the above equation into the domain D.*

Proof The identities

$$F(a,b;c;z) = F(a,b+1;c+1;z) - \frac{a(c-b)}{c(c+1)} z F(a+1,b+1;c+2;z),$$
$$F(a,b+1;c+1;z) = F(a+1,b+1;c+2;z) - \frac{(b+1)(c-a+1)}{(c+1)(c+2)} z F(a+1,b+2;c+3;z)$$

may be derived straightforwardly by definition. Therefore, if we set

$$P_{2n} = F(a+n, b+n; c+2n; z), \quad n = 0, 1, 2, \ldots,$$
$$P_{2n+1} = F(a+n, b+n+1; c+2n+1; z), \quad n = 0, 1, 2, \ldots,$$

we see that the three-term recursion relation

$$P_n = P_{n+1} + a_{n+1} z P_{n+2}$$

holds. Note also that

$$\lambda(a_n z) = 1, \quad \lambda(b_n) = \lambda(1) = 0,$$

and

$$\lambda(P_n/P_{n+1}) = 0, \quad n = 1, 2, 3, \ldots.$$

From (34), we have

$$\lim_{k\to\infty} a_n = -\frac{1}{4}.$$

Therefore, by Theorem 5, the regular C-fraction

$$1 + \operatorname*{K}_{n=1}^{\infty}\left(\frac{a_n z}{1}\right) = 1 + \frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots$$

corresponds to $f(z)$ in (37). □

Since $F(a, 0; c; z) = 1$, by setting $b = 0$, and replacing c by $c-1$ in (37), the C-fraction representation of $F(a, 1; c; z)$ may be obtained.

Corollary 7 *Let a and c be complex constants such that $\{a_n\}$ defined by*

$$a_{2n+1} = -\frac{(a+n)(c+n-1)}{(c+2n-1)(c+2n)}, \quad n = 0, 1, 2, \ldots,$$
$$a_{2n} = -\frac{n(c-a+n-1)}{(c+2n-2)(c+2n-1)}, \quad n = 1, 2, 3, \ldots, \tag{38}$$

is a sequence of non-zero complex numbers. Then,

(A) For all z such that $|z| < 1$,

$$F(a,1;c;z) = \frac{1}{1 + \operatorname*{K}_{n=1}^{\infty}\left(\frac{a_n z}{1}\right)}$$
$$= \frac{1}{1} - \frac{\frac{a}{c}z}{1} - \frac{\frac{1(c-a)}{c(c+1)}z}{1} - \frac{\frac{(a+1)c}{(c+1)(c+2)}z}{1} - \frac{\frac{2(c-a+1)}{(c+2)(c+3)}z}{1} - \frac{\frac{(a+2)(c+1)}{(c+3)(c+4)}z}{1} - \cdots. \tag{39}$$

(B) The continued fraction on the right-hand side of (39) converges to a function $f(z)$ meromorphic in the domain D of (36), and $f(z)$ is the analytic continuation in D of $F(a, 1; c; z)$. $f(z)$ is holomorphic at $z = 0$, and $f(0) = 1$. The continued fraction converges uniformly on compact subsets of D.

Note that the C-fraction (39) is equivalent to

$$F(a,1;c;z)=\frac{1}{1}\,\frac{}{-}\,\frac{az}{c}\,\frac{}{-}\,\frac{1(c-a)z}{c+1}\,\frac{}{-}\,\frac{(a+1)cz}{c+2}\,\frac{}{-}\,\frac{2(c-a+1)z}{c+3}\,\frac{}{-}\,\frac{(a+2)(c+1)z}{c+4}\,\frac{}{-}\,\cdots,$$

which follows from Theorem 1.

The confluent hypergeometric function, also called the Kummer function, is defined by

$$M(a,c;z)=1+\frac{a}{c}\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\ldots, \tag{40}$$

where a and c are complex constants with $c \notin [0,-1,-2,\ldots]$. It is also denoted as ${}_1F_1(a;c;z)$.

Theorem 8 *Let $\{a_n\}$ be a sequence of complex numbers defined by*

$$\begin{aligned} a_{2n} &= \frac{a+n}{(c+2n-1)(c+2n)}, \quad n=1,2,3,\ldots,\\ a_{2n+1} &= -\frac{c-a+n}{(c+2n)(c+2n+1)}, \quad n=0,1,2,\ldots, \end{aligned} \tag{41}$$

where a and c are complex constants chosen such that

$$a_n \neq 0, \quad n=1,2,3,\ldots.$$

Then:

(A) The regular C-fraction

$$1+\operatorname*{K}_{n=1}^{\infty}\left(\frac{a_n z}{1}\right) \tag{42}$$

converges to the meromorphic function

$$f(z)=\frac{M(a;c;z)}{M(a+1;c+1;z)}$$

for all $\mathbb{C}$.

(B) The convergence is uniform on every compact subset of $\mathbb{C}$ which contains no poles of $f(z)$.

(C) $f(z)$ is holomorphic at $z=0$, and $f(0)=1$.

Proof Let

$$\begin{aligned} P_{2n} &= M(a+n;c+2n;z),\\ P_{2n+1} &= M(a+n+1;c+2n+1;z), \quad n=0,1,2,\ldots. \end{aligned}$$

Then it can be shown that P_n satisfy the three-term recurrence relations

$$P_n = P_{n+1}+a_{n+1}zP_{n+2}, \quad n=0,1,2,\ldots.$$

Since

$$\lambda(a_{n+1}z)=1, \quad \lambda(a)=0, \quad \lambda(P_n/P_{n+1})=0, \quad n=0,1,2,\ldots,$$

by Theorem 4, the C-fraction (42) corresponts to P_0/P_1. Furthermore, since

$$\lim_{n\to\infty} a_n = 0,$$

the statements (B) and (C) follow. □

Setting $b=0$ and replacing $c+1$ by c in (41), we obtain the C-fraction representation of $M(1;c;z)$.

Corollary 9 *Let c be a complex number such that* $\{a_n\}$ *defined by*

$$\begin{aligned} a_{2n} &= \frac{n}{(c+2n-2)(c+2n-1)}, \\ a_{2n+1} &= -\frac{c+n-1}{(c+2n-1)(c+2n)}, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{43}$$

is a sequence of non-zero complex numbers. Then

$$M(1;c;z) = \frac{1}{1-}\,\frac{\frac{1}{c}z}{1+}\,\frac{\frac{1}{c(c+1)}z}{1-}\,\frac{\frac{c}{(c+1)(c+2)}z}{1+}\,\frac{\frac{2}{(c+2)(c+3)}z}{1-}\,\frac{\frac{c+1}{(c+3)(c+4)}z}{1+}\cdots. \tag{44}$$

Note that the C-fraction (44) is equivalent to

$$M(1;c;z) = \frac{1}{1-}\,\frac{z}{c+}\,\frac{1\cdot z}{c+1-}\,\frac{cz}{c+2+}\,\frac{2\cdot z}{c+3-}\,\frac{(c+1)z}{c+4}\,\frac{3\cdot z}{+c+5-}\cdots.$$

9 Emergence of Power Law Decay out of Exponential Decay

The recursion relation (7) may be written as

$$\frac{L_n}{Z_n} = \cfrac{1}{1+\cfrac{\frac{Z_n}{Y_{n+1}}}{1+\frac{L_{n+1}}{Y_{n+1}}}} = \frac{1}{1+}\,\frac{\frac{Z_n}{Y_{n+1}}}{1}\,\frac{\frac{L_{n+1}}{Y_{n+1}}}{+\;1}$$

Thus, the continued fraction L_0 for the modified Schiessel-Blumen model shown in Figure 4 becomes

$$\frac{L_0}{Z_0} = \frac{1}{1+}\,\frac{\frac{Z_0}{Y_1}}{1+}\,\frac{\frac{Z_1}{Y_1}}{1+}\,\frac{\frac{Z_1}{Y_2}}{1+}\,\frac{\frac{Z_2}{Y_2}}{1+}\cdots = \frac{1}{1+}\,\frac{\frac{k_1}{c_0 s}}{1+}\,\frac{\frac{k_1}{c_1 s}}{1+}\,\frac{\frac{k_2}{c_1 s}}{1+}\,\frac{\frac{k_2}{c_2 s}}{1+}\cdots$$

with $Z_0 = 1/c_0 s$. By choosing an appropriate unit of time t, we can set $c_0 = 1$. Let $\{a_n\}$ be the sequence such that

$$\begin{aligned} a_{2n+1} &= \frac{c_n}{k_{n+1}}, \quad n = 0, 1, 2, \ldots, \\ a_{2n} &= \frac{c_n}{k_n}, \quad n = 1, 2, 3, \ldots, \end{aligned} \tag{45}$$

then we obtain

$$L_0 = \frac{1}{s}\left(\frac{1}{1+}\,\frac{a_1/s}{1+}\,\frac{a_2/s}{1+}\,\frac{a_3/s}{1+}\cdots\right). \tag{46}$$

Thus, if we choose a_n as given by (38), we obtain

$$L_0(s) = \frac{1}{s}F\left(a, 1; c; -\frac{1}{s}\right)$$

Using the Doetch symbol to denote the Laplace transform $F(s)$ of $f(t)$ as $F(s) \bullet\!\!-\!\!\circ f(t)$, we have

$$\frac{1}{s^n} \bullet\!\!-\!\!\circ \frac{t^{n-1}}{(n-1)!}, \quad n > 0. \tag{47}$$

By applying this relation to the defining equations (33) and (40), it can be shown that

$$\frac{1}{s}F\left(a, 1; c; -\frac{1}{s}\right) \bullet\!\!-\!\!\circ M(a; c; -t).$$

Thus, the impulse response of the modified Schiessel-Blumen model is given by

$$x(t) = M(a; c; -t). \tag{48}$$

For large t, we have the asymtotic expansion [11]

$$x(t) = \frac{\Gamma(c)}{\Gamma(c-a)} t^{-a}\left(1 + O(|t|^{-1})\right),$$

and hence if a is a fractional number, we have the power-law decay $x(t) \sim t^{-a}$ with a fractional exponent.

10 Examples

As a typical example, let us choose $a = \frac{1}{2}$ and $c = 1$ in (34) and (38). Then, $a_1 = \frac{1}{2}$, and $a_n = \frac{1}{4}$, $n = 2, 3, 4, \ldots$, and the $(2n+1)$-th approximant f_{2n+1} gives the n loop approximation of the modified Schiessel-Blumen model (46). We list some of the approximants explicitly.

$$\begin{aligned}
f_1 &= \frac{1}{s} \bullet\!\!-\!\!\circ\ 1 \\
f_3 &= \frac{1+4s}{3s+4s^2} \bullet\!\!-\!\!\circ\ \frac{1}{3} + \frac{2}{3} e^{-\frac{3}{4}t} \\
f_5 &= \frac{1+12s+16s^2}{5s+20s^2+16s^3} \bullet\!\!-\!\!\circ\ \frac{1}{5} + \frac{2}{5}\left(e^{-\frac{1}{8}(5+\sqrt{5})t} + e^{-\frac{1}{8}(5-\sqrt{5})t}\right) \\
f_7 &= \frac{1+24s+80s^2+64s^3}{7s+56s^2+112s^3+64s^4} \bullet\!\!-\!\!\circ\ \ldots \\
f_9 &= \frac{1+40s+240s^2+448s^3+256s^4}{9s+120s^2+432s^3+576s^4+256s^5} \bullet\!\!-\!\!\circ\ \ldots \\
&\ \vdots \\
L_0 &= \frac{1}{s} F\left(\frac{1}{2}, 1; 1; -\frac{1}{s}\right) \bullet\!\!-\!\!\circ\ M\left(\frac{1}{2}; 1; -t\right)
\end{aligned} \tag{49}$$

Note that the inverse Laplace transform of f_{2n+1} for $n \geq 3$ may not be expressed in terms of exact numbers. It is nevertheless clear that they are sums of exponential functions since f_{2n+1} are rational functions of s.

In order to produce figures of such results, we use *MATHEMATICA*® 5.0. We first define the sequence $\{a_n\}$ defined in (38),

```
In[1]:= an[a_, c_, n_?OddQ] := ((a + (n - 1) / 2) (c + (n - 1) / 2 - 1)) / ((c + (n - 1) - 1) (c + (n - 1)));
         an[a_, c_, n_?EvenQ] := ((n / 2) (c - a + n / 2 - 1)) / ((c + n - 2) (c + n - 1));
         an[a_, c_, 1] := a / c;
         an[a_, c_, 0] := 1;
```

and the function which generates the continued fraction as

```
In[5]:= f[x_] := Append[Drop[x, -2], Last[Drop[x, -1]] / (1 + Last[x])]
In[6]:= cFraction[a_List] := First[Nest[f, a, Length[a] - 1]]
```

The n-loop approximation of the impedance $L_0(s)$ is obtained by the inverse Laplace transform of the $(2n+1)$-th approximant of the C-fraction,

```
In[7]:= Loops[n_, t_] :=
          InverseLaplaceTransform[Simplify[cFraction[Take[seq, 2 n + 1] / s]], s, t]
```

These are essentially all the definitions that we need. For producing plots, it is convenient to define another function,

```
In[8]:= PlotSequence[seq_, opt___] := Module[{p1, p2},
          p1 = ListPlot[seq, PlotStyle → PointSize[0.015], DisplayFunction → Identity];
          p2 = ListPlot[seq, PlotJoined → True, DisplayFunction → Identity]; Show[p1, p2, opt,
           PlotRange → {0, 1}, AxesLabel → {"n", "aₙ"}, DisplayFunction → $DisplayFunction]]
```

and load the package,

```
In[9]:= Needs["Graphics`Graphics`"]
```

which is needed for log-log plots. We can now see that the function `cFraction` generates the continued fraction, e.g.,

```
In[10]:= cFraction[{a1, a2, a3, a4}]
```

Out[10]= $\dfrac{a1}{1+\frac{a2}{1+\frac{a3}{1+a4}}}$

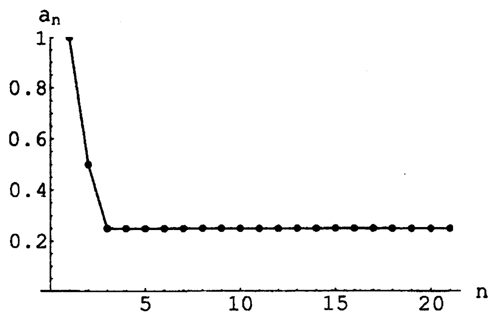
Case 1

With $a = \frac{1}{2}$ and $c = 1$, which yield the result (49), the first 21 terms of $\{a_n\}$ are

In[11]:= `seq = Table[an[`$\frac{1}{2}$`, 1, n], {n, 0, 20}]`

Out[11]= $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$

```
In[12]:= PlotSequence[seq];
```



The impulse response $x(t)$ of two loops with five elements of the modified Schiessel-Blumen ladder is given by the inverse Laplace transform of f_5.

```
In[13]:= Loops[2, t]
```

Out[13]= $\frac{1}{5} + \frac{2}{5} e^{-\frac{1}{8}(5+\sqrt{5})t}\left(1 + e^{\frac{\sqrt{5}t}{4}}\right)$

In this way, we can plot the impulse response $x(t)$ from one to six loops,

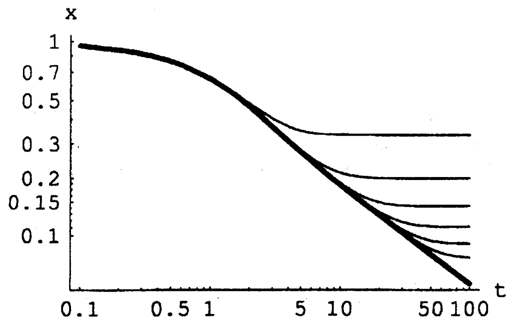
```
In[14]:= Do[p[n] = LogLogPlot[Evaluate[Loops[n, t]], {t, 0.1, 100}], {n, 1, 6}];
```

Here, we do not show the plot. In the limit of infinite loops, $x(t)$ approaches $M(\frac{1}{2}, 1, -t)$, which is plotted as

In[15]:= `p[0] = LogLogPlot[Hypergeometric1F1[`$\frac{1}{2}$`, 1, -t],`
`{t, 0.1, 100}, PlotStyle → Thickness[0.01], AxesLabel → {"t", "x"}];`

Here, we show all the curves together,

```
In[16]:= Show[Table[p[n], {n, 0, 6}]];
```



As n increases, $x(t)$ approaches $M\left(\frac{1}{2}, 1, -t\right)$ shown by the thick line. The straight line implies its power-law decay $\sim t^{-1/2}$, and the slope corresponds to its exponent $-1/2$.

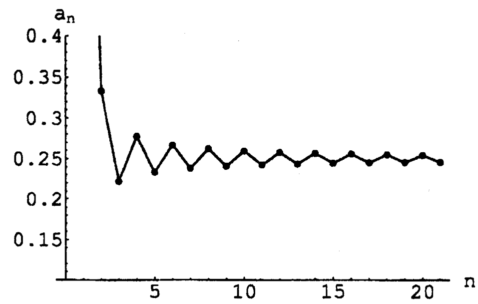
Case 2

For $a = \frac{2}{3}$ and $c = 2$, we can repeat the same computations.

```
In[17]:= seq = Table[an[2/3, 2, n], {n, 0, 20}]
```

Out[17]= $\left\{1, \frac{1}{3}, \frac{2}{9}, \frac{5}{18}, \frac{7}{30}, \frac{4}{15}, \frac{5}{21}, \frac{11}{42}, \frac{13}{54}, \frac{7}{27}, \frac{8}{33}, \frac{17}{66}, \frac{19}{78}, \frac{10}{39}, \frac{11}{45}, \frac{23}{90}, \frac{25}{102}, \frac{13}{51}, \frac{14}{57}, \frac{29}{114}, \frac{31}{126}\right\}$

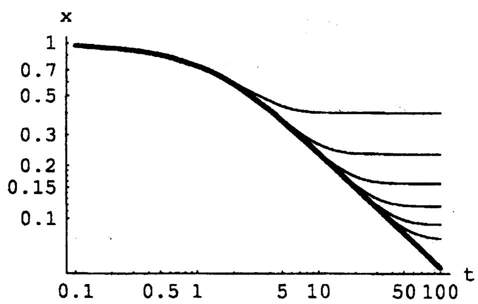
```
In[18]:= PlotSequence[seq];
```




```
In[19]:= Do[p[n] = LogLogPlot[Evaluate[Loops[n, t]], {t, 0.1, 100}], {n, 1, 6}];

In[20]:= p[0] = LogLogPlot[Hypergeometric1F1[2/3, 2, -t],
            {t, 0.1, 100}, PlotStyle → Thickness[0.01], AxesLabel → {"t", "x"}];

In[21]:= Show[Table[p[n], {n, 0, 6}]];
```



Again, as n increases, $x(t)$ approaches $M\left(\frac{2}{3}, 2, -t\right)$. Again, the straight line implies the power-law decay with the exponent $-2/3$.

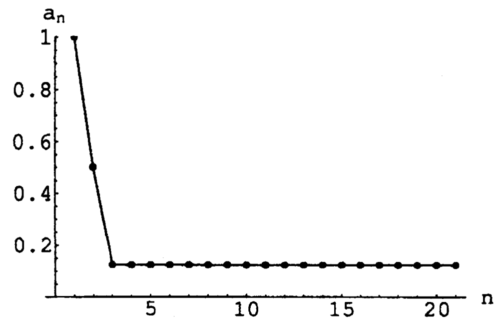
Case 3

We do not know the general form of $x(t)$ other than the case with $\{a_n\}$ given by (38). But for some finite loops, we could try any sequence in the same way as above. For example, let us consider the sequence

```
In[22]:= seq = Join[{1, 1/2}, Table[an[1/2, 1, n]/2, {n, 2, 20}]]

Out[22]= {1, 1/2, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8}

In[23]:= PlotSequence[seq];
```

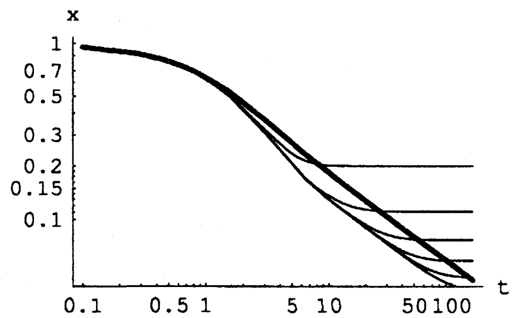


which is obtained by $a_0 = 1, a_1 = \frac{1}{2}$, and $\{\frac{1}{2}a_n\}, n = 2, 3, \ldots$, where $\{a_n\}$ is the sequence (38) with $a = \frac{1}{2}$ and $c = 1$. The corresponding impulse response becomes

```
In[24]:= Do[p[n] = LogLogPlot[Evaluate[Loops[n, t]],
           {t, 0.1, 150}, PlotPoints → 50, PlotRange → All], {n, 1, 6}];

In[25]:= p[0] = LogLogPlot[Hypergeometric1F1[1/2, 1, -t],
           {t, 0.1, 150}, PlotStyle → Thickness[0.01], AxesLabel → {"t", "x"}];

In[26]:= Show[Table[p[n], {n, 0, 6}]];
```



Here, $M\left(\frac{1}{2}, 1, -t\right)$ is plotted together just for reference. It is amusing to speculate that, as n increases, $x(t)$ approaches a straight line in the log-log plot, but in fact we were unable to prove it. If this is indeed the case, we could estimate the exponent a of $x(t) = t^{-a}$ for large t by plotting curves with more approximations.

Case 4

If we choose the sequence $\{a_n\}$ as given by (43), then we have

$$L_0(s) = \frac{1}{s} M\left(1; c; -\frac{1}{s}\right).$$

Again, by applying (47), we can show that

$$\frac{1}{s} M\left(1; c; -\frac{1}{s}\right) \bullet\!\!-\!\!\circ \; {}_0F_1(c; -t)$$

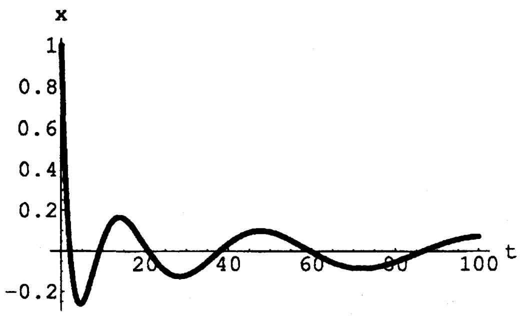
where

$$ {}_0F_1(c;z) = 1 + \frac{1}{c}\frac{z}{1!} + \frac{1}{c(c+1)}\frac{z^2}{2!} + \frac{1}{c(c+1)(c+2)}\frac{z^3}{3!} + \dots $$

is the confluent hypergeometric function. Hence the inverse Laplace transform of $L_0(s)$ is given by

$$ x(t) = {}_0F_1(c : -t) $$

```
In[27]:= Plot[Hypergeometric0F1[4/3, -t], {t, 0, 100}, PlotRange -> All,
         PlotStyle -> Thickness[0.01], AxesLabel -> {"t", "x"}];
```



According to (45), the sequence (43) corresponds to the alternating sign for k_n and c_n. Negative values of these parameters imply anti-damping, i.e., the oscillation of the system will be amplified, as is expected by the above plot.

11 Conclusions

The impulse response $x(t)$ of the modified Schissel-Blumen ladder in Figure 4 shows relaxation that obeys the power-law decay. In fact, we have shown that $x(t)$ is explicitly given by the hypergeometric function, as in (48), when the parameters obey the relation (45) with (38). We remark that the relation (45) may be rewritten as

$$ \frac{k_{n+1}}{k_n} = \frac{a_{2n}}{a_{2n+1}}, \quad \frac{c_{n+1}}{c_n} = \frac{a_{2n+2}}{a_{2n+1}}. $$

Therefore, from (38) it follows that $k_{n+1}/k_n \to 1$ and $c_{n+1}/c_n \to 1$ as $n \to \infty$. Thus the strengths of the springs and dampers in the substructure of the modified Schiessel-Blumen ladder are all the same for large ladders. This establishes the fact that the power-law decay is a characteristic property of the underlying fractal structure of the ladder.

Since not much is known about the asymptotic expansion of continued fractions, we were unable to identify the general structure of fractional power-law in a more general form. It is expected from the examples, however, that the power-law decay would emerge in a wide class of parameter relations.

It is also shown that if some of the spring constants and damping coefficients have negative values, the impulse response may be oscillatory. This is consistent since negative values of such parameters imply amplifying effects of the oscillators.

Finally, we remark that the successive approximations of the continued fraction expansion of the modified Schiessel-Blumen ladder is reminiscent of the successive approximations of the Taylor series. In fact, the Taylor series expansion of e^{-t} is given by

$$ e^{-t} = \sum_{k=0}^{\infty} \frac{1}{k!}(-t)^k = 1 + \frac{1}{1!}(-t) + \frac{1}{2!}(-t)^2 + \frac{1}{3!}(-t)^3 + \dots. $$

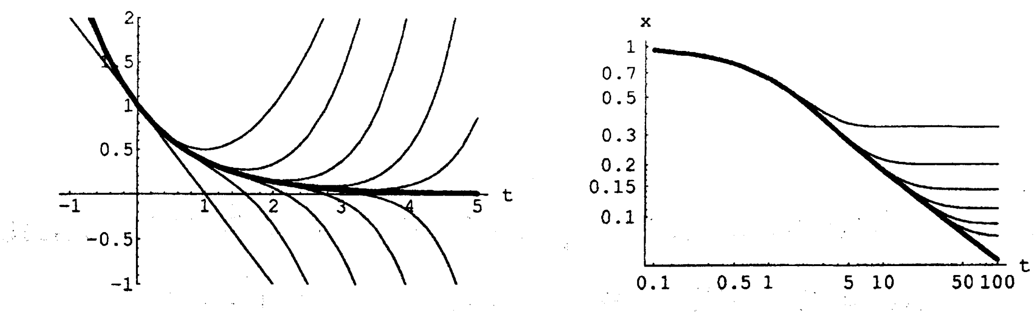


Figure 5: Successive approximations of the Taylor expansion of e^{-t} and Schiessel-Blumen ladder

The exponential decay is successively approximated by polynomials, which have power-law behavior. On the other hand, in our result

$$t^{-a} \approx M(a;c;-t) \circ\!\!-\!\!\bullet\, L_0(s) = \frac{1/s}{1+\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{a_n/s}{1}\right)} = \frac{1}{s}\left(\frac{1}{1+}\,\frac{a_1/s}{1}\,+\,\frac{a_2/s}{1}\,+\,\cdots\right),$$

the power-law decay is successively approximated by the inverse Laplace transform of the continued fraction expansion. It is amusing to compare these cases in the figure shown in Figure 5.

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